Indian Statistical Institute

Final Examination, Fall 2021

Analysis of Several Variables, M.Math First Year

Time : 3 Hours Date : 05.01.2022 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Q1. (10 marks) Prove that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$. Also compute the quotient group

$$GL_n(\mathbb{R})/SL_n(\mathbb{R}).$$

Q2. (15 marks) If $A = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$, then compute $exp(A)$.

Q3. (15 marks) Consider the function

$$f(x) = \begin{cases} \frac{x^3 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that the directional derivative of f exists in every direction at $(0,0)$. Also, examine the differentiability of f at $(0,0)$.

Q4. (15 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Prove that f is not injective.

Q5. (15 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Use Fubini's theorem to prove that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Q6. (15 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function. Compute $f^*(dx \wedge dy)$.

Q7. (20 marks) Let ω be a nonzero differential form on an open set $\mathcal{O} \subseteq \mathbb{R}^n$. Prove that there exists a chain c such that

$$\int_c \omega \neq 0.$$

Use this fact, Stokes' theorem and $\partial^2 = 0$ to prove that $d^2 = 0$.